# THE PRINCIPLE OF HAMILTON-OSTROGRADSKII FOR SYSTEMS WITH ONE-SIDED CONSTRAINTS

(PRINTSIP GAMIL'TONA-OSTROGRADSKOGO DLIA SISTEM S ODNOSTORONNIMI SVIAZIAMI)

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Holonomic systems with two-sided constraints represent a well-known section of analytical mechanics whereas holonomic systems with one-sided (or nonholding, releasing) constraints have not been sufficiently investigated. In this paper the Hamilton-Ostrogradskii integral variational principle is established for holonomic systems with one-sided constraints and is proved to be necessary and sufficient.

1. We shall consider a holonomic mechanical system under the action of potential forces. We shall assume that the generalized coordinates $q_1, \ldots, q_m, q_{m+1}, \ldots, q_n$ are chosen such that the one-sided constraints applied to the system are determined as follows:

$$q_{m+1} \geqslant 0, \ldots, q_n \geqslant 0 \tag{1.1}$$

Such a choice of the generalized coordinates is always possible.

We shall consider the real path of the system $q_1(t), \ldots, q_n(t)$ during the interval of time $[t_0, T]$. In the most general bases this path can consist of sections of $r+1$ different types, where $r$ is the number of different combinations of the indices $m+1, \ldots, n$ which is obviously equal to $2^{n-m}$.

The type of the motion on a section, depends on which of the coordinates (1.1) are equal to zero on that section. We shall assign an index to each possible type of the sections of motion, where the index zero is given to the section in which all $q_k > 0$ $(k = m+1, \ldots, n)$. To each number $\alpha$ corresponds the totality $J_\alpha$ of the indices $m+1, \ldots, n$, such that if $k \in J_\alpha$, $q_k = 0$ on each given section of type $\alpha$.

It is assumed that during the interval of time $[t_0, T]$ the system goes from one section of motion to another a finite number ($N$) of times. Let $L = T + U$ be a Lagrange function. The equations of motion of the system on a section of type $\alpha$ have the form

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \qquad \begin{pmatrix} j = 1, \ldots, m \\ j \text{ does not belong to } J_\alpha \end{pmatrix} \\
&\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \lambda_k \qquad (k \in J_\alpha) \\
&q_k = 0 \qquad (k \in J_\alpha)
\end{aligned} \tag{1.2}$$

with the condition

$$\lambda_k \geqslant 0 \qquad (k \in J_\alpha) \tag{1.3}$$

where a dot represents differentiation with respect to $t$.

The transition from one section of motion to another can be of two types: with or without an impulse. An impulse can occur, if a system which did not have some one-sided constraint again acquires one. For instance, let the system pass at time $t_s$ from a section of type zero to a section of the first type for which $q_{m+1} = 0$. If $\dot{q}_{m+1}(t_s) \geqslant 0$, there won't be an impulse; on the other hand if $\dot{q}_{m+1}(t_s) < 0$, there will be an impulse [1]: the velocities $\dot{q}_i$ $(i = 1, \ldots, n)$ have a discontinuity whereupon $\dot{q}_{m+1}$ takes on instantaneously a nonnegative value.

Thus, when we pass from one section of motion of type $\alpha$ to another of type $\beta$ the variations of the velocities $\dot{q}_i$ $(i = 1, \ldots, n)$ can be continuous or have a discontinuity. If at the instant of the impulse the system leaves the constraint $q_k = 0$, $k \in J_\alpha - J_\alpha \times J_\beta$ and acquires the constraint $q_k = 0$, $k \in J_\beta - J_\alpha \times J_\beta$, then at the instant of the impulse (*)

$$\delta q_k = 0 \qquad (k \in J_\beta) \tag{1.4}$$

The values of the velocities after the impulse satisfy the equations of the impulse theory

$$\left(\frac{\partial L}{\partial \dot{q}_j}\right)_0 = \left(\frac{\partial L}{\partial \dot{q}_j}\right)_1 \qquad \begin{pmatrix} j = 1, \ldots, m \\ j \text{ does not belong to } J_\beta \end{pmatrix} \tag{1.5}$$

The indices 0 and 1 in (1.5) show that in $L$ we have the values of the velocities before and after the impulse.

We shall assume that the constraints applied at the time of the impulse are retained, then there is a sufficient number of Equations (1.5) to determine the velocities after the impulse. If this were not true, it would be necessary to make complementary assumptions on the behavior of the system after the impulse [2].

Together with the real path, we shall consider the totality of the devious paths, consisting of configurations permissible by the constraints and infinitely close to the real path [3]

$$q_1(t) + \delta q_1(t), \ldots, q_n(t) + \delta q_n(t)$$

where the $\delta q_j$ $(j = 1, \ldots, m)$ and those of the $\delta q_k$ $(k = m+1, \ldots, n)$, for which on the real path $q_k > 0$, are arbitrarily chosen, and $\delta q_k \geqslant 0$ for the $q_k$ which are zero on the real path. Among the totality of the devious paths we shall choose those which coincide with the real path at instants $t_0$ and $T$ in such a way that

$$\delta q_i(t_0) = 0, \qquad \delta q_i(T) = 0 \qquad (i = 1, \ldots, n) \tag{1.6}$$

Let's consider the Hamiltonian action integral for the interval $[t_0, T]$

$$S = \int_{t_0}^{T} L dt \tag{1.7}$$

We shall compute the increment of the action $S$ when we pass from the real path to a devious path with the linear approximation accuracy with respect to $\delta q_i$, $\delta \dot{q}_i$, i.e. we compute the variation

$$\delta S = \delta \int_{t_0}^{T} L dt = \delta \sum_{s=0}^{N+1} \int_{t_s}^{t_{s+1}} L\, dt \tag{1.8}$$

where the $t_1, \ldots, t_N$ are the time instants at which the transitions from one section of motion to the other occur, whereupon $t_{N+1} = T$. Since the instants at which the system passes from one section of motion to another can be varied, we should apply to the integrals in (1.8) the formula of the nonsynchronous variation [3]

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*) $A \times B$ denotes the intersection of the ensemble $A$, $B$.

$$\delta S = \sum_{s=0}^{N+1} \left( \int_{t_s}^{t_{s+1}} \delta L\, dt + L|_{t=t_{s+1}} \delta t_{s+1} - L|_{t=t_s} \delta t_s \right)$$

However

$$\sum_{s=0}^{N+1} (L|_{t=t_{s+1}} \delta t_{s+1} - L|_{t=t} \delta t_s) = 0$$

since the time $(T - t_0)$ is fixed. We have

$$\delta S = \sum_{s=0}^{N+1} \int_{t_s}^{t_{s+1}} \left[ \sum_{i=1}^{n} \left( \frac{\partial L}{\partial q_i^{\cdot}} \delta q_i^{\cdot} + \frac{\partial L}{\partial q_i} \delta q_i \right) \right] dt$$

Integrating by parts we get

$$\delta S = \sum_{s=0}^{N+1} \sum_{i=1}^{n} \frac{\partial L}{\partial q_i^{\cdot}} \delta q_i \Big|_{t_s}^{t_{s+1}} + \sum_{s=0}^{N+1} \sum_{i=1}^{n} \int_{t_s}^{t_{s+1}} \left( \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial q_i^{\cdot}} \right) \delta q_i \, dt$$

The first term of this sum is equal to zero on the basis of Equations (1.4), (1.6) and impulse equations (1.5).

Taking into consideration, that the sections of the motion have different types, we get

$$\delta S = - \sum_{\alpha=0}^{r} \sum_{s=0}^{N+1} \sum_{k \in J_\alpha} \int_{t_s}^{t_{s+1}} \lambda_k \delta q_k \, dt$$

Since $\lambda_k \geqslant 0$, $\delta q_k \geqslant 0$ $(k \in J_\alpha)$, then

$$\delta S \leqslant 0 \qquad (1.9)$$

If the real path on all its sections is comparable to devious paths of similar types, i.e. if the conditions

$$\delta q_k = 0 \qquad (k \in J_\alpha) \qquad (1.10)$$

are fulfilled, we find $\delta S = 0$.

Thus the necessity of the following principle is proved. The first variation of the Hamiltonian action integral is nonpositive, if the devious paths coinciding with the real path at the initial and final instants of time $t_0$ and $T$ and for which the conditions (1.1) are fulfilled are comparable to the real path. The Hamiltonian action integral has a stationary value if we can also compare to the real path the devious paths which satisfy the condition (1.10).

In order to prove that this principle is sufficient we shall deduce from it the equations of motion. The problem consists in finding the necessary conditions for obtaining an extremum of the functional (1.7) when conditions (1.1) are fulfilled. This is a problem with one-sided variations.

Applying to the given case a theorem from [4] generalized to an $(n+1)$ dimensional space, we get the following result.

T h e o r e m 1.1 . If a curve $\Gamma\{q_i(t)(i = 1, \ldots, n)\}$ which gives an extremal value to the integral (1.7) among the curves $q_i(t)(i = 1, \ldots, n)$, belonging to the closed domain (1.1), and connecting two given points and such that the $q_i^{\cdot}(t)$ $(i = 1, \ldots, n)$ are continuous, except, may be, at the points $A_\nu$, where $\Gamma$ goes from a domain $\Phi_0\{q_k > 0 \quad (k = m+1, \ldots, n)\}$ to the boundary and from the boundary $\Phi_\alpha\{q_k = 0,\ k \in J_\alpha\}$ to the boundary $\Phi_\beta\{q_k = 0,\ k \in J_\beta\}$, then

1) the pieces of the curve $\Gamma$, belonging to the domain $\Phi_0$ are extremal for the integral $S$, i.e. satisfy Euler equations

$$\frac{d}{dt} \frac{\partial L}{\partial q_i^{\cdot}} - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, \ldots, n)$$

2) The pieces of the curve belonging to the boundary $\Phi_\alpha$, yield the extrema of the problem on the arbitrary extremum $S$ for the conditions $q_k = 0$, $k \in J_\alpha$, i.e. satisfy Equations (1.2). Furthermore, at each point of the boundary, in the case of a nonpositive first variation $\delta S$, the condition (1.3) must be satisfied;

3) At the points where $\Gamma$ passes from the domain $\Phi_0$ to the boundary or from the boundary $\Phi_\alpha$ to the boundary $\Phi_\beta$, the condition (1.5) and the relations

$$(L)_1 - (L)_0 - \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i^{\cdot}}\right)_0 (q^{\cdot}_{i1} - q^{\cdot}_{i0}) = 0$$

are valid.

If we eliminate the case in which the points $A_\nu$ are corner points of the sought curve, the condition (3) can be replaced by the following condition: the curve $\Gamma$ possesses a continuously rotating tangent, i.e. the $q^{\cdot}_i(t)$ $(i = 1, \ldots, n)$ are continuous functions of $t$. Thus the sufficiency is proved.

N o t e . Let us investigate the nature of the extremum of the action $S$ for systems with one-sided constraints. It can be stated, that if the real path of the system in the interval $(t_0, T)$ consists of sections belonging to the domain $\Phi_0$ and of the boundaries $\Phi_\alpha, \Phi_\beta, \ldots$, then, on such a path there will not be any local maxima nor minima of the action $S$. (We consider that the sections which lie on the boundaries $\Phi_\alpha, \Phi_\beta, \ldots$, are not extremas of the integral $S$, since if the contrary were true the constraints (1.1) could be completely ignored).

In fact, by comparing the real path with the devious paths for which condition (1.10) is fulfilled, we find that $\delta S = 0$. Furthermore, by virtue of the positive-definiteness of the kinetic energy, the Legendre condition of positiveness of the second order variation of the motion $\delta^2 S$ is always fulfilled. This proves that on the real path the maximum of $S$ is never obtained. On the other hand, comparing the real path and the devious paths which coincide with the real path in the domain $\Phi_0$ and for which on the boundaries $\Phi_\alpha, \Phi_\beta, \ldots$ $\delta q_k > 0$, $k \in J_\alpha$, $k \in J_\beta, \ldots$ we get $\delta S < 0$.

Thus, it is possible to choose devious paths satisfying the conditions (1.1) and (1.6) such that the value of the action $S$ is not smaller than on the real path, and others such that the value of the action $S$ is smaller than on the real path.

**2.** We have considered above the variational principle of Hamilto-Ostrogradskii valid only for holonomic systems with potential forces. However, in mechanics this principle has a more general meaning. It is also applicable to systems with nonpotential forces. The principle consists in the fact that on the real path which is composed, as before, of sections of different types, the integral

$$\delta' R = \int_{t_0}^{T} \left[ \delta T + \sum_{\nu=1}^{n} (X_\nu \delta x_\nu + Y_\nu \delta y_\nu + Z_\nu \delta z_\nu) \right] dt, \qquad T = \frac{1}{2} \sum_{\nu=1}^{n} m_\nu (m_\nu^{\cdot 2} + y_\nu^{\cdot 2} + z_\nu^{\cdot 2})$$

is not positive for any values $\delta x_\nu, \delta y_\nu, \delta z_\nu$, infinitely close to the $x_\nu(t)$, $y_\nu(t)$, $z_\nu(t)$, which correspond to the real path and become equal to zero for $t_0$ and $T$ which determine in the interval $(t_0, T)$ the motions which are admissible by one-sided constraints. This integral is equal to zero for the nondisengaging motions [2] (fulfillment of the condition (1.10)). Here $x_\nu, y_\nu, z_\nu$ $(\nu = 1, \ldots, n)$ are the orthogonal coordinates of the points of the mechanical system; $T$ is the kinetic energy; $X_\nu, Y_\nu, Z_\nu$ are the acting forces; $m_\nu$ the mass of the $\nu$th point.

Let us prove this principle. We notice that

$$\delta T = \sum_{\nu=1}^{n} m_\nu (x_\nu^{\cdot} \delta x_\nu + y_\nu^{\cdot} \delta y_\nu + z_\nu^{\cdot} \delta z_\nu)$$

Taking into account the presence of $N$ sections of motion and integrating by parts, we bring $\delta' R$ to the form

$$\delta' R = \sum_{s=0}^{N+1} \sum_{\nu=1}^{n} (m_\nu x_\nu^{\cdot} \delta x_\nu + m_\nu y_\nu^{\cdot} \delta y_\nu + m_\nu z_\nu^{\cdot} \delta z_\nu) \Big|_{t_s}^{t_{s+1}} +$$

$$+ \sum_{s=0}^{N+1} \int_{t_s}^{t_{s+1}} \left\{ \sum_{\nu=1}^{n} [(X_\nu - m_\nu x_\nu^{\cdot\cdot}) \delta x_\nu + (Y_\nu - m_\nu y_\nu^{\cdot\cdot}) \delta y_\nu + (Z_\nu - m_\nu z_\nu^{\cdot\cdot}) \delta z_\nu] \right\} dt$$

The first term of this sum is equal to zero since the $\delta x_\nu$, $\delta y_\nu$, $\delta z_\nu$ are equal to zero for $t_0$ and $T$ and on the basis of the general impulse theory which, in the present case, when the only impulses acting on the system are the constraint impulses, has the form [2]

$$\sum_{\nu=1}^{n} [\Delta (m_\nu x_\nu^{\cdot}) \delta x_\nu + \Delta (m_\nu y_\nu^{\cdot}) \delta y_\nu + \Delta (m_\nu z_\nu^{\cdot}) \delta z_\nu] = 0$$

where the $\Delta (m_\nu x_\nu^{\cdot})$, $\Delta (m_\nu y_\nu^{\cdot})$, $\Delta (m_\nu z_\nu^{\cdot})$ are the differences between the values $m_\nu x_\nu^{\cdot}$, $m_\nu y_\nu^{\cdot}$, $m_\nu z_\nu^{\cdot}$ before and after the impulse. From d'Alembert-Lagrange's principle for systems with one-sided constraints

$$\sum_{\nu=1}^{n} [(X_\nu - m_\nu x_\nu^{\cdot\cdot}) \delta x_\nu + (Y_\nu - m_\nu y_\nu^{\cdot\cdot}) \delta y_\nu + (Z_\nu - m_\nu z_\nu^{\cdot\cdot}) \delta z_\nu] \leqslant 0$$

(here the sign of the equality is valid for the nondisengaging possible motions), we get $\delta' R \leqslant 0$ and $\delta' R = 0$ for the nondisengaging possible motions which was to be proved.

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